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# ON REFLECTIONS IN EUCLIDEAN SPACES GENERATING FREE PRODUCTS

BY

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I. Let $\varphi$ and $\psi$ be two rotations in three-dimensional Euclidean space $R^3$ through angles $\pi$ and $2\pi/3$ radians, the axes of which intersect in an angle $\vartheta$.

Moreover, if $\alpha$ is an element of an abstract group, let $(\alpha)$ denote the group generated by $\alpha$.

Then we can formulate a theorem of F. HAUSDORFF ([3] p. 469–472) as follows:

If $\cos \vartheta$ is transcendental, $\varphi$ and $\psi$ generate a free product of the groups $(\varphi)$ and $(\psi)$.

It easily follows that a suitable countable set of elements of HAUSDORFF's group forms a set of free generators of a free group. (see e.g. W. SIERPIŃSKI [4] p. 236).

J. DE GROOT [2] proved that two rotations about perpendicular axes through equal angles $\alpha$ generate a free non-abelian group, if $\cos \alpha$ is transcendental. Moreover he constructed a free rotation group of continuous rank. (Another one may also be obtained from a lemma of W. SIERPIŃSKI [4]).

Furthermore we mention some theorems of S. ŚWIERCZKOWSKI [5], which induced our results.

1. Two rotations about perpendicular axes rotating through an angle arccos $\frac{1}{3}$ generate a free non-abelian group.
2. A series of neatly piled tetrahedra (i.e. two consecutive tetrahedra have one common face and three consecutive ones are distinct) does not contain two tetrahedra which are congruent by translation.

For related literature we refer to the list in T. J. DEKKER [1].

A new aspect of ŚWIERCZKOWSKI's method is that the elements of his group are represented by matrices of algebraic numbers.

We shall prove that the reflections in the faces of a regular $n$-dimensional simplex ($n \geqslant 3$) generate a free product of cyclic groups (Theorem I).

If barycentric coordinates are chosen, these reflections are represented by matrices of rational numbers.

The result of S. Świerczkowski about regular tetrahedra is extended to higher dimensional regular simplices in theorem III, which the author has announced in „Stelling" VII, joined to his thesis [1].

In a simple way free non-abelian rotation groups in $R^n$ ($n \geqslant 3$) are obtained in theorem IV.

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II. Let $T^n$ ($n \geqslant 3$) be a regular $n$-dimensional simplex situated in $R^n$. Moreover let $\omega_i$ ($i = 0, \ldots, n$) denote the reflections in the ($n - 1$)-dimensional faces of $T^n$. Obviously the $\omega_i$ are of order two. We shall prove now

Theorem I. *The group generated by the reflections* $\omega_i$ ($i = 0, \ldots, n$) *is a free product of the cyclic groups* $(\omega_i)$.

Proof. Let the points of $R^n$ be represented by barycentric coordinates $(x_0, \ldots, x_n)$ with respect to $T^n$. Then the reflection $\omega_i$ in the face $x_i = 0$ ($i = 0, \ldots, n$) is represented by the matrix $(a^i_{kl})$, the elements being given by:

$$\begin{aligned} a^i_{kk} &= 1 && \text{if} \quad k \neq i \\ a^i_{ii} &= -1 && \\ a^i_{ki} &= 2/n && \text{if} \quad k \neq i \\ a^i_{kl} &= 0 && \text{if} \quad l \neq i \quad \text{and} \quad l \neq k. \end{aligned}$$

Let $G^n$ denote the group generated by the $\omega_i$ ($i = 0, \ldots, n$). The elements of $G^n$ may obviously be written in the form

$$\alpha = \omega_{i_1}\omega_{i_2} \ldots \omega_{i_s}, \tag{1}$$

where two consecutive $\omega_i$ do not cancel. Let $\alpha$ be represented by the matrix $(a_{kl})$ and $\beta = \alpha\omega_i$ by the matrix $(b_{kl})$. Putting $a = 2/n$, we have obviously

$$\begin{aligned} b_{kl} &= a_{kl} \quad \text{if} \quad l \neq i, \\ b_{ki} &= -a_{ki} + a \textstyle\sum_{l \neq i} a_{kl}. \end{aligned} \tag{2}$$

By induction with respect to $s$ we find that the elements $a_{kl}$ are polynomials in $a$ with integral coefficients, and moreover with first coefficients $\pm$ 1 if $l = i_s$ and of lower degree than $a_{ki_s}$ or identically zero if $l \neq i_s$.

This is obviously true for $s = 1$.

Furthermore if the statement holds for the elements $a_{kl}$, then according to (2) — notice that $i \neq i_s$, since $\omega_i$ and $\omega_{i_s}$ do not cancel — the leading term of $b_{ki}$ equals $a$ times the leading term of $a_{ki_s}$, from which easily the statement follows for the elements $b_{kl}$.

Since $n \geqslant 3$ and $a = 2/n$, the polynomials in $a$ do not vanish, thus $\alpha$ is unequal to identity, which completes the proof.

Theorem II. *The group generated by the reflections* $\omega_i$ $(i = 0, \ldots, n)$ *does not contain translations apart from the identity.*

Proof. Suppose $\alpha \in G^n$ is a translation unequal to identity. Then $\beta = \omega_k \alpha \omega_k$ is also a translation, thus $\alpha$ and $\beta$ are commutative. Obviously there is an $\omega_k$ such that $\omega_k$ and the extreme factors of $\alpha$ do not cancel. Hence we find a non-trivial relation of the form

$$\omega_k \alpha \omega_k \alpha \omega_k \alpha^{-1} \omega_k \alpha^{-1} = 1,$$

contrary to theorem I.

Theorem III. *If* $T_1^n, \ldots, T_s^n$ $(n \geqslant 3, s > 1)$ *are such regular n-dimensional simplices situated in* $R^n$ *that*

1. $T_j^n$ *and* $T_{j+1}^n$ *have strictly one common* $(n - 1)$*-dimensional face* $(j = 1, \ldots, s - 1)$;
2. $T_j^n \neq T_{j+2}^n$ $(j = 1, \ldots, s - 2)$;

*then* $T_1^n$ *and* $T_s^n$ *are not congruent by translation.*

Proof. Obviously we have for some $i$ $(0 \leqslant i \leqslant n)$

$$T_2^n = \omega_i T_1^n.$$

Thus by induction with respect to $s$ we find

$$T_s^n = \alpha T_1^n,$$

where $\alpha$ has the form (1).

Suppose there is a translation $\tau$ such that

$$T_s^n = \tau T_1^n.$$

Then $\tau^{-1}\alpha$ is an isometry which exchanges the vertices of $T_1$. Thus $\tau^{-1}\alpha$ is of finite order, hence for some $m$ the isometry $\alpha^m$ is a translation belonging to $G^n$, contrary to theorem II.

III. Let $\tau_i$ be a translation in the direction orthogonal to the hyperplane $x_i = 0$ $(i = 0, \ldots, n)$. Then obviously $\tau_i \omega_i$ is a reflection in a hyperplane parallel to the hyperplane $x_i = 0$.

Since the translation group is a normal subgroup of the group of isometries, theorem II immediately yields that the group generated by the reflections $\tau_i\omega_i$ $(i = 0, \ldots, n)$ is a free product of the cyclic groups $(\tau_i\omega_i)$. Obviously the translations $\tau_i$ may be chosen in such a way that the reflections $\tau_i\omega_i$ $(i = 0, \ldots, n)$ have a common fixed point $p$. Then the group generated by them may clearly be considered as a transformation group of an $(n-1)$-dimensional sphere $S^{n-1}$ with centre $p$ or of the infinitive hyperplane, which is an $(n-1)$-dimensional elliptic space $E^{n-1}$.

Let $\varphi_i = \tau_i\omega_i$ $(i = 0, \ldots, n)$ be a set of reflections leaving some point $p$ fixed. Then obviously the rotations $\alpha = \varphi_0\varphi_1$ and $\beta = \varphi_2\varphi_3$ are free generators of a free non-abelian group of rotations about $p$.

Let $p$ be the origin of a Cartesian coordinate-system. Then we have

Theorem IV. *The group of rotations in $R^n$ $(n \geqslant 3)$ about the origin contains a free non-abelian subgroup, freely generated by the rotations $\alpha$ and $\beta$.*

In particular, if $n = 3$, $\alpha$ and $\beta$ rotate through an angle $2 \arccos \frac{1}{3}$ and the axes are perpendicular. In fact, the group generated by $\alpha$ and $\beta$ is a subgroup of ŚWIERCZKOWSKI's group (see [5]).

It easily follows that the rotations

$$\alpha^i\beta\alpha^{-i} \quad (i = 0, 1, 2, \ldots)$$

are free generators of a free group.

From these results a free rotation group of continuous rank may be obtained in the same way as in [2].

IV. It is clear, that the theorems I and II also hold for many irregular $n$-dimensional simplices $(n \geqslant 3)$. In particular we have:

If $T^n$ $(n \geqslant 3)$ is an $n$-dimensional simplex in $R^n$ with the property that the lengths of the edges have algebraically independent proportions and if moreover $\omega_i$ $(i = 0, \ldots, n)$ are reflections in the $(n-1)$-dimensional faces of $T^n$, then the theorems I and II are valid.

## LITERATURE


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